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SYNTHESIS OF A CONTROL IN THE PROBLEM OF THE TIME-OPTIMAL TRANSFER OF A POINT MASS TO A SPECIFIED POSITION WITH ZERO VELOCITY†

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The problem of the time-optimal control of the motion of a point mass by means of a force of bounded modulus is considered. It is required that the point be transferred from an arbitrary state of motion to the origin of the system of coordinates with zero velocity. By introducing self-similar conjugate variables, the solution of the two-point problem can be successfully reduced to a search for the optimal root of a certain function, specified analytically. A complete solution of the control problem in the form of a synthesis is obtained using mathematical modelling methods. The feedback coefficients along the unit vectors of the position and velocity vectors are found and a control algorithm and a Bellman function are constructed. Examples using practical initial data are presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

The controlled motions of a point mass of constant mass under the action of a bounded force are investigated. The point has to be transferred from its arbitrary current position to the origin of coordinates of a certain inertial system with a zero final velocity ("a soft landing").

The controllable system, the terminal conditions, the constraints and the functional are described by relations of the form

$$\dot{x}=v,\quad \dot{v}=u;\quad x(0)=x^0,\quad v(0)=v^0$$
$$x(t_f)=0,\quad v(t_f)=0;\quad t_f\to\min_u,\quad |u|\leqslant 1 \tag{1.1}$$

Here, x, v, u are vectors of arbitrary dimension n, $n \geqslant 2$ and x^0, v^0 are the (initial) data, obtained by measurements. The somewhat more general case of arbitrary constant values of the mass m and a constraint on the controlling force $|u| \leqslant u_0$ is reduced to a problem of the form (1.1) by means of elementary substitutions. The case of an arbitrary final value of the variable x^f and initial value of the time t_0 is treated in a similar manner.

We also note that system (1.1) possesses central (spherical) symmetry and the general case of a dimensionality $n \geqslant 2$ of the geometrical space is equivalent to that of a plane ($n = 2$, see below). When $x^0 \neq 0$, $v^0 \neq 0$, $|(x^0, v^0)| < |x^0||v^0|$, the plane is specified by these vectors. In the case of equality, that is, $x^0 = 0$ or $v^0 = 0$, $|(x^0, v^0)| = |x^0||v^0|$, the problem degenerates and becomes one-dimensional. The situation of a general position is considered next.

The formulation of the time-optimal problem (1.1) is quite standard and simple. Methods for the optimal control of motion in the form of the maximum principle [1], dynamic programming [1, 2], the l-problem of moments [3], as well as direct variational and numerical methods [4, 5] can be used. However, no solution of the problem has yet been obtained. The problem of synthesis is of special interest, namely, to construct the optimal feedback controls $u_s = u^*(x, v)$ and Bellman function $t_f = T(x, v)$ which is the least time to transfer a point to the state (0, 0) from a position x, v at instant of time t (which remains until landing). Particular results are available: the classical problem for a one-dimensional system $n = 1$, see [1]; the case when the value of $v(t_f)$ is not specified (a "hard landing"), see [1, 6, 7] and others. It is well known that the solution of the boundary-value problem for arbitrary initial conditions and the investigation of the analytic properties of the Bellman function and the optimal control present fundamental difficulties [1, 2, 6, 7]. As is customary in the case of control problems, these functions are piecewise-smooth with respect to the phase variables x and v.

We shall first apply the necessary conditions for optimality in the method of dynamic programming

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[1, 2, 5] to problem (1.1). In the case of a Bellman function $T = T(x, \upsilon)$, we obtain a functional-differential equation of the form

$$(T'_x, \upsilon) + \min_u (T'_\upsilon, u) = -1, \quad |u| \leq 1 \tag{1.2}$$

The corresponding derivatives are denoted by primes and the minimum of the scalar product is taken with respect to the vector u which takes values from a unit n-dimensional sphere. In addition, the function T must be strictly positive when $x, \upsilon \neq 0$ and vanish when $x = \upsilon = 0$.

On carrying out the operation of minimization in (1.2), we arrive at a Cauchy problem for a non-linear partial differential equation (the Hamilton–Jacobi–Bellman equation) [1, 2, 5, 8] of the form

$$\begin{gathered} (p, \upsilon) + |q| = 1, \quad p = -T'_x, \quad q = -T'_\upsilon; \quad u^* = q|q|^{-1} \\ T = T(x, \upsilon) > 0, \quad |x| + |\upsilon| > 0; \quad T(0, 0) = 0 \end{gathered} \tag{1.3}$$

The solution of problem (1.3) is sought in the class of piecewise-smooth functions.

We note that the natural condition $T > 0$ is necessary since absurd results may be obtained if it is ignored. Actually, the function $T = (e, \upsilon)$, which is linear with respect to υ and where e is a constant unit n-vector ($|e| = 1$), satisfies all the relations (1.3) (apart from $T > 0$), but certainly it is not the Bellman function of the problem.

Considerable difficulties, due to the smoothlessness of the Bellman function T, arise when using dynamic programming to solve the synthesis problem [1, 2]. This function undergoes discontinuities of the first kind in certain manifolds of lower dimensionality and its derivatives will be generalized (singular) functions. We shall apply the optimality conditions in the form of the maximum principle [1].

We introduce the variables (momenta) which are conjugate to the phase variables x and υ, respectively. The solution of the time-optimal problem reduces to constructing the solution of the two-point boundary-value problem

$$\begin{gathered} \dot{x} = \upsilon, \quad \dot{\upsilon} = u^*, \quad u^* = q|q|^{-1}; \quad x(0) = x^0, \quad \upsilon(0) = \upsilon^0, \quad x(t_f) = 0 \\ \upsilon(t_f) = 0, \quad p(t) = p^0 = \text{const}, \quad q(t) = -p^0 t + q^0; \quad q^0 = \text{const} \end{gathered} \tag{1.4}$$

where p and q can be normalized. System (1.4) can be completely integrated in terms of elementary functions. The permissible values of the unknowns p^0, q^0 and $t_f > 0$ which satisfy the final conditions for x and υ have to be determined.

2. CONSTRUCTION OF THE PHASE TRAJECTORY

We will now write an explicit analytic expression for the phase trajectory $x(t)$, $\upsilon(t)$. Using the structure of the optimal control $u^*(t)$ and substituting it into the equations of motion, using (1.4), we obtain, after integration, a representation for the velocity vector $\upsilon(t)$ in terms of elementary functions of the form

$$\begin{gathered} \upsilon(t) = \upsilon^0 + \int_0^t \frac{Q(\tau)}{R(\tau)} d\tau = \upsilon^0 + \frac{1}{\rho^2}[-\xi R(\tau) + (\rho\eta - \sigma\xi)V(\tau)]_0^t \\ Q(t) = -\xi t + \eta, \quad R(t) = |Q(t)| = (\rho^2 t^2 - 2\sigma\rho t + 1)^{1/2} \\ V(t) = \operatorname{arsh} \varkappa, \quad \varkappa = (\rho t - \sigma)(1 - \sigma^2)^{-1/2}, \quad \rho = |\xi|, \quad \eta = q^0 |q^0|^{-1} \\ \xi = p^0 |q^0|^{-1}, \quad |\eta| = 1, \quad \operatorname{arsh} \varkappa = \ln[\varkappa + (1 + \varkappa^2)^{1/2}] \\ (\xi, \eta) = \rho\sigma, \quad \sigma = \cos(\xi, \eta), \quad -1 \leq \sigma \leq 1; \quad u^* = Q(t)/R(t) \end{gathered} \tag{2.1}$$

The functions Q and R in (2.1) are defined for all t, ξ η and V is odd with respect to $\varkappa$ since $-\varkappa + (1 + \varkappa^2)^{1/2} = [\varkappa + (1 + \varkappa^2)^{1/2}]^{-1}$. Note that, in (2.1), the normalization mentioned earlier is carried out on the quantity $|q^0|$. As a result, the function $\upsilon(t)$ depends on time explicitly, the known (measured) n-vector of the parameters υ^0 and, also, on the $2n - 1$ unknown ξ and η, which have to be determined from the boundary conditions (1.1) (when $t = t_f$, where t_f is also unknown). The parameters σ, ρ are defined in terms of the n-vectors ξ, η, where $|\eta| = 1$.

Using the representation $\upsilon(t)$ found in (2.1), we obtain, using the formula for repeated integration, a similar elementary representation for the position vector $x(t)$

$$x(t) = x^0 + \upsilon^0 t + \int_0^t \frac{(t-\tau)Q(\tau)}{R(\tau)} d\tau = x^0 + t\upsilon(t) + \\ -\frac{\xi}{2\rho^3}[(\rho\tau + 3\sigma)R(\tau) + (3\sigma^2 - 1)V(\tau)]_0^t - \frac{\eta}{\rho^2}[\sigma V(\tau) + R(\tau)]_0^t \tag{2.2}$$

The functions $\upsilon(t)$, $R(t)$, $Y(t)$ are defined by (2.1). The upper value $\tau = t$ and the lower value $\tau = 0$ of τ are not substituted as yet in (2.1), (2.2) and the subsequent equations for brevity. Explicit representations of the required phase variables $x(t)$, $\upsilon(t)$ are thereby obtained using elementary algebraic and logarithmic functions. For convenience in the subsequent constructions, they can be represented in the form of "linear expressions" in the vectors ξ, η as follows:

$$\upsilon(t) = \upsilon^0 + V_\xi \xi + V_\eta \eta, \quad V_\xi = -\rho^{-2}(\sigma V + R)\Big|_0^t, \quad V_\eta = \rho^{-1} V\Big|_0^t$$
$$x(t) = x^0 + \upsilon^0 t + X_\xi \xi + X_\eta \eta, \quad X_\eta = \rho^{-2}[-R + (\rho\tau - \sigma)V]_0^t \tag{2.3}$$
$$X_\xi = \frac{1}{2\rho^3}[(-\rho\tau + 3\sigma)R + (-2\rho\sigma\tau + 3\sigma^2 - 1)V]_0^t$$

Here, $V_{\xi,\eta}(t)$, $X_{\xi,\eta}(t)$ are known scalar functions of t which also depend on the unknown parameters ρ, σ; they vanish when $t = 0$.

To avoid misunderstandings, it should be noted that the dependence of x and υ on ξ and η will actually be extremely non-linear since the coefficients $X_{\xi,\eta}$, $V_{\xi,\eta}$ contain algebraic and transcendental (logarithmic) functions of ρ and σ. These functions are fairly smooth for the values of the argument $t \geqslant 0$ and the parameters $\rho \geqslant 0$, $|\sigma| < 1$ being considered. A separate treatment is required when $|\sigma| = 1$ (see Section 4).

3. REDUCTION OF THE SYSTEM OF BOUNDARY CONDITIONS

The efficient construction of the solution of boundary-value problem (1.4), that is, the determination of the unknown ξ and η and the minimum value of t_f for arbitrary values of x^0, υ^0, presents the main difficulty in solving the time-optimal problem (1.1). We have a system of $2n$ linear algebraic equations in the vectors ξ and η, obtained from (2.3) using (1.4): $x(t_f) = 0$, $\upsilon(t_f) = 0$, that is

$$X_\xi(t_f)\xi + X_\eta(t_f)\eta = -x^0 - \upsilon^0 t_f \\ V_\xi(t_f)\xi + V_\eta(t_f)\eta = -\upsilon^0 \tag{3.1}$$

System (3.1) is uniquely solvable for arbitrary x^0, υ^0, t_f, ρ, σ since it is established by the Bunyakovskii–Cauchy inequality that its determinant is non-zero

$$X_\xi(t_f)V_\eta(t_f) - V_\xi(t_f)X_\eta(t_f) \neq 0$$

System (3.1) has a partitioned diagonal matrix and the required ξ and η are found from it in an elementary manner (as in the case of scalar ξ and η) in the form of linear functions of x^0, υ^0

$$\xi = \xi^*(x^0, \upsilon^0, t_f, \rho, \sigma) \equiv \xi_x^* x^0 + \xi_\upsilon^* \upsilon^0 \\ \eta = \eta^*(x^0, \upsilon^0, t_f, \rho, \sigma) \equiv \eta_x^* x^0 + \eta_\upsilon^* \upsilon^0 \tag{3.2}$$

Here, $\xi_{x,\upsilon}^*(t_f, \rho, \sigma)$, $\eta_{x,\upsilon}^*(t_f, \rho, \sigma)$ are scalar functions of the unknown (t_f, ρ, σ). Using elementary scalar-product operations, we obtain a system of three transcendental equations in the unknown (t_f, ρ, σ)

$$\eta^{*2} = \eta_x^{*2} x^{02} + 2\eta_x^* \eta_\upsilon^* |x^0||\upsilon^0| c + \xi_\upsilon^{*2} \upsilon^{02} = 1$$
$$\xi^{*2} = \xi_x^{*2} x^{02} + 2\xi_x^* \xi_\upsilon^* |x^0||\upsilon^0| c + \xi_\upsilon^{*2} \upsilon^{02} = \rho^2 \tag{3.3}$$
$$(\xi^*, \eta^*) = \xi_x^* \eta_x^* x^{02} + (\xi_x^* \eta_\upsilon^* + \xi_\upsilon^* \eta_x^*) |x^0||\upsilon^0| c + \xi_\upsilon^* \eta_\upsilon^* \upsilon^{02} = \rho\sigma$$

Here, c is an unknown scalar parameter which is determined in the same way as σ (see (2.1)) by scalar multiplication of the vectors x^0, υ^0: $(x^0, \upsilon^0) = |x^0||\upsilon^0|c$. The unknowns t_f, ρ, σ are arguments of the elementary transcendental functions $\xi^*_{x,\upsilon}$, $\eta^*_{x,\upsilon}$, see (2.1)–(2.3) and (3.1)–(3.2).

It is interesting to note that system (3.3) is determined by the three parameters $|x^0||\upsilon^0|c$, which characterize the vectors x^0, υ^0 in the plane specified by these vectors ($|c| < 1$). The simple degenerate case when $c = \pm 1$ has been studied in [1, 2]. It corresponds to a one-dimensional system, that is, to motion of the object (a point mass) along the line joining the geometrical point x^0 to the origin of coordinates.

Note that system (3.3) is linear in the known (measured) parameters (x^0, υ^0) and can be solved for them. These parameters are determined as functions of the unknown $t_f > 0$, $\rho > 0$, $-1 \leqslant \sigma \leqslant 1$.

The final solution of system (3.3) can be obtained by numerical and numerical-graphical methods. However, a "direct" approach seems to be difficult to achieve, due to the high dimension of system (3.3) (it is equal to three). The introduction of self-similar variables and a reduction in the dimensionality of the system being solved can considerably reduce the computational costs, and the required solution can be obtained in a clear numerical-graphical form [7, 8].

An analysis of expressions (2.1) and (2.2) shows that it is preferable to introduce the vector $\zeta = t_f\xi$ instead of ξ; its modulus $|\zeta| = \mu = t_f\rho$. The vector equations in ζ and η, which are analogous to Eqs (3.1) for ξ and η, then become

$$x^0 = t_f^2(a_\zeta \zeta + a_\eta \eta), \quad \upsilon^0 = t_f(b_\zeta \zeta + b_\eta \eta)$$
$$a_\zeta = a_\zeta(\mu, \sigma) = -(1/2\mu^3)[(\mu + 3\sigma)a + \mu + (3\sigma^2 - 1)b]$$
$$a_\eta = a_\eta(\mu, \sigma) = \mu^{-2}(a + \sigma b), \quad b_\zeta = b_\zeta(\mu, \sigma) = \mu^{-2}(a + \sigma b) \tag{3.4}$$
$$b_\eta = b_\eta(\mu, \sigma) = b/\mu, \quad a = a(\mu, \sigma) \equiv R|_0^{t_f} = (\mu^2 - 2\mu\sigma + 1)^{1/2} - 1$$
$$b = b(\mu, \sigma) \equiv V|_0^{t_f} = \operatorname{arsh}(\mu - \sigma)(1 - \sigma^2)^{-1/2} + \operatorname{arsh}\sigma(1 - \sigma^2)^{-1/2}$$

Note that $a = b = 0$ when $\mu = 0$ and the coefficients $a_{\zeta,\eta}$, $b_{\zeta,\eta}$ have finite limits when $\mu \to 0$ if $\sigma \neq \pm 1$. This property immediately follows from expressions (3.4) or it can be established from (2.1) and (2.2) when $t = t_f$ by changing the variable of integration $\tau = t_f\theta$ (the upper limit for θ will then be $\theta_f = 1$).

We will now analyse the vector equations (3.4) which relate the known (measured) quantities x^0, υ^0 and the unknown ζ, η, t_f. Unlike what was described above (see (3.2)) we shall not solve them for ζ and η. By analogy with (3.3), by scalar product operations we obtain a system of three equations for determining the unknown parameters t_f, μ, σ

$$x^{02} = t_f^4(a_\zeta^2\mu^2 + 2a_\zeta a_\eta \mu\sigma + a_\eta^2) \equiv t_f^4 f_x^2(\mu, \sigma) \geqslant 0$$
$$\upsilon^{02} = t_f^2(b_\zeta^2\mu^2 + 2b_\zeta b_\eta \mu\sigma + b_\eta^2) \equiv t_f^2 f_\upsilon^2(\mu, \sigma) \geqslant 0 \tag{3.5}$$
$$|x^0||\upsilon^0|c = t_f^3[a_\zeta b_\zeta \mu^2 + (a_\zeta b_\eta + a_\eta b_\zeta)\mu\sigma + a_\eta b_\eta] \equiv t_f^3 f_{x\upsilon}(\mu, \sigma)$$

The important difference between systems (3.5) and (3.4) lies in the fact that the introduction of the parameter $\mu = \rho t_f$ enables one to separate out the unknown t_f and to obtain a defining system of two equations in the unknown μ and σ. This system is non-uniquely constructed depending on the values of $l = |x^0|$, $h = |\upsilon^0|$.

4. DETERMINATION OF THE SOLUTION OF THE BOUNDARY-VALUE PROBLEM

If the magnitude of the velocity h is sufficiently large, it is preferable to use the equation

$$l^2h^{-4} \equiv \varphi_v^2 = f_x^2(\mu, \sigma)f_v^{-4}(\mu, \sigma), \quad clh^{-3} \equiv \psi_v = f_{xv}(\mu, \sigma)f_v^{-3}(\mu, \sigma) \tag{4.1}$$

to determine the unknowns μ and σ. Permissible roots (μ_i, σ_i), $\mu_i \geqslant 0$, $|\sigma_i| \leqslant 1$ have to be found which produce a minimum value of the functional t_f. From (3.5), we obtain

$$t_f^* = \min_i t_{fi}, \quad t_{fi} = t_f(h, \mu_i, \sigma_i), \quad t_f = h|f_v(\mu, \sigma)|^{-1}$$
$$\mu_i = \mu_i(\varphi_v^2, \psi_v^2), \quad \sigma_i = \sigma_i(\varphi_v^2, \psi_v^2) \tag{4.2}$$

Here, φ_v^2, ψ_v^2 are known (measured) quantities which are determined on the basis of l, h, and c from (4.1). It follows from (4.2) that the root (μ_i, σ_i) which produces a maximum of the function $|f_v(\mu_i, \sigma_i)|$ will be the optimal root.

If, however, the distance l is sufficiently large, it is necessary to use the relations

$$h^4l^{-2} \equiv \varphi_x^2 = f_v^4(\mu, \sigma)f_x^{-2}(\mu, \sigma), \quad \mathrm{ch}\,l^{-1/2} \equiv \psi_x = f_{xv}(\mu, \sigma)|f_x(\mu, \sigma)|^{-3/2}$$
$$t_{fi} = l^{1/2}|f_x(\mu_i, \sigma_i)|^{-1/2} \to \min_i, \quad \mu_i = \mu_i(\varphi_x, \psi_x), \quad \sigma = \sigma_i(\varphi_x, \psi_x) \tag{4.3}$$

which are the inverse of (4.1), to determine μ and σ.

We find from (4.3) that the root (μ_i, σ_i), which produces a maximum value of $|f_x|$ will be the optimal root. Note that φ_v^2 and φ_x^2 are functionally related: $\varphi_v^2 = \varphi_x^{-2}$. From an applied point of view, relations (4.1)–(4.3) are inconvenient since the quantities $\varphi_{v,x}$, $\psi_{v,x}$ can be quite large ("unbounded") when $l/h^2 \to \infty$ or $h^2/l \to \infty$ which involves considerable computing costs. The computational difficulties can be reduced by introducing the normalized quantities $\Phi_{x,v}$

$$\Phi_x^2 \equiv \frac{l^2}{l^2+h^4} = f_x^2(\mu, \sigma)F^{-2}(\mu, \sigma) \equiv L^2(\mu, \sigma)$$
$$\Phi_v^2 \equiv \frac{h^4}{l^2+h^4} = f_v^4(\mu, \sigma)F^{-2}(\mu, \sigma) \equiv H^2(\mu, \sigma)$$
$$\Psi \equiv clh(l^2+h^4)^{-3/4} = f_{xv}(\mu, \sigma)F^{-3/2}(\mu, \sigma) \tag{4.4}$$
$$F^2(\mu, \sigma) \equiv f_x^2(\mu, \sigma) + f_v^4(\mu, \sigma), \quad 0 \leqslant \Phi_{x,v}^2 \leqslant 1$$
$$t_f = (l^2+h^4)^{1/4}|F(\mu, \sigma)|^{-1/2}, \quad t_f^* = \min_i t_f(\mu_i, \sigma_i)$$

Here $\Phi_{x\,v}^2$, Ψ are measurable (specified) and calculable quantities. Note that Φ_x^2 and Φ_v^2 are functionally related: $\Phi_x^2 + \Phi_v^2 = 1$ (as previously φ_x^2, φ_v^2). The unknown (μ, σ) are determined by a pair of independent equations for Φ_x^2, Ψ or Φ_v^2, Ψ. Since the expression for Ψ is unbounded when $l, h \to 0$, it is preferable to use the equation

$$c = f_{xv}(\mu, \sigma)|f_x(\mu, \sigma)|^{-1}|f_v(\mu, \sigma)|^{-1} \equiv C(\mu, \sigma), \quad |c| \leqslant 1 \tag{4.5}$$

By the Bunyakovski–Cauchy integral inequality, the right-hand side C in (4.5) has a modulus which does not exceed unity for all permissible values of $\mu \geqslant 0$, $|\sigma| \leqslant 1$.

Hence, it is next necessary to solve a particular pair of equations: $\Phi_x^2 \equiv L^2$, $c = C$ or $\Phi_v^2 \equiv H^2$, $c = C$ for (μ, σ) and to choose the optimal root (μ_i, σ_i). It follows from (4.4) that this root corresponds to a maximum of the function $|F(\mu, \sigma)|$

$$(\mu^*, \sigma^*) = (\mu_i, \sigma_i)^* = \arg\max_i |F(\mu_i, \sigma_i)| \tag{4.6}$$

Having determined the optimal values of (μ^*, σ^*) and t_f^* it is possible to construct the controls and trajectories.

5. CONSTRUCTION OF THE OPTIMAL CONTROL

Suppose the optimal values of μ^*, σ^*, t_f^* have been determined. Then, by (2.1), the optimal control can be constructed in the form of a program and a synthesis. According to the substitution $\zeta = t_f\xi$ made in Section 3 (see (3.4)), the expression for the programmed control is transformed in the following manner

$$u_p^* = Q^*(t)/|Q^*(t)|; \quad Q^*(t) = -\zeta^*(t/t_f^*) + \eta^*, \quad |\zeta|^* = \mu^*$$
$$|Q^*(t)| = (\mu^{*2}(t/t_f^*)^2 - 2\sigma^*\mu^*(t/t_f^*) + 1)^{1/2}, \quad |\eta^*| = 1 \tag{5.1}$$

The optimal values of the n-vectors ζ^*, η^* in (5.1) are determined by Eqs (3.4), the solution of which is constructed in a similar way to the solution of (3.2). This solution can be found in an elementary manner since the matrix of the system consists of four $(n \times n)$-blocks, each of which is proportional to the unit matrix. As a result, we obtain the expressions

$$\zeta^* = \alpha_x x^0 + \alpha_v v^0, \quad \alpha_x = b_\eta^* / (t_f^{*2}\delta^*), \quad \alpha_v = -a_\eta^* / (t_f^*\delta^*)$$
$$\eta^* = \beta_x x^0 + \beta_v v^0, \quad \beta_x = -b_\zeta^* / (t_f^{*2}\delta^*), \quad \beta_v = -a_\zeta^* / (t_f^*\delta^*) \tag{5.2}$$
$$\delta^* = \alpha_\zeta^* b_\eta^* - a_\eta^* b_\zeta^*, \quad a_{\zeta,\eta}^* = a_{\zeta,\eta}(\mu^*, \sigma^*), \quad b_{\zeta,\eta}^* = b_{\zeta,\eta}(\mu^*, \sigma^*)$$

The functions $a_{\zeta,\eta}$, $b_{\zeta,\eta}$ are defined by (3.4), where μ^*, σ^* are solely dependent on two parameters, c and L^2 (or H^2), for example.

In expressions (5.2) it is possible to get rid of the quantity t_f^* and to represent them in the form

$$\zeta^* = (b_\eta^* / \delta^*) F^* \Phi_x n_x^0 - (a_\eta^* / \delta^*)(F^* \Phi_v)^{1/2} n_v^0$$
$$\eta^* = -(b_\zeta^* / \delta^*) F^* \Phi_x n_x^0 + (a_\zeta^* / \delta^*)(F^* \Phi_v)^{1/2} n_v^0 \tag{5.3}$$
$$n_x^2 = x^0 / l^0, \quad |n_x^0| = 1, \quad n_v^0 = v^0 / h, \quad |n_v^0| = 1$$

Substituting the expressions for ζ^*, η^* of the form (5.2) or (5.3) into (5.1), we obtain the required optimal control in the form of a program which depends on t, x^0, v^0 and the three parameters of motion l, h, c

$$u_p^* = u_p^*(t, x^0, v^0, l, h, c), \quad l = |x^0|, \quad h = |v^0|, \quad c = \cos(x^0, v^0) \tag{5.4}$$

The optimal time $t_f^* = t_f^*(l, h, c)$ is determined using (4.2)–(4.4), (4.6) and the optimal trajectories are defined by expressions (2.3) in which $\xi = \xi^* = \zeta^*/t_f^*$, $\rho = \rho^* = \mu^*/t_f^*$, $\sigma = \sigma^*$. As a result, the problem of programmed optimal control in the case of fixed x^0, v^0 is completely solved.

We will now consider the problem of constructing a time-optimal synthesis. We assume that the roots μ^*, σ^* and the value of t^*_f have been determined as functions of the variables $l = |x|$, $h = |v|$, $c = \cos(x, v)$ from a sufficiently wide range of values of $(l, h) \in D \subset R^2$ and $-1 \leqslant |c| \leqslant 1$. Then, using (5.1), we obtain the feedback control u^*_s and the Bellman function T of the problem

$$u_s^* = u_s^*(l, h, c, x, v) = \eta^*(l, h, c, x, v)$$
$$\eta^* = -(b_\zeta^* / (T^2\delta^*))x + (a_\zeta^* / (T\delta^*))v \equiv k_x x + k_v v =$$
$$= -(b_\zeta^* / \delta^*) F^* \Phi_x n_x + (a_\zeta^* / \delta^*)(F^* \Phi_v)^{1/2} n_v \equiv K_x n_x + K_v n_v$$
$$\mu^* = \mu^*(|x|, |v|, (x, v)|x|^{-1}|v|^{-1}), \quad \sigma^* = \sigma^*(|x|, |v|, (x, v)|x|^{-1}|v|^{-1}) \tag{5.5}$$
$$n_x = x|x|^{-1}, \quad n_v = v|v|^{-1}$$
$$T(x, v) = t_f^*(|x|, |v|, (x, v)|x|^{-1}|v|^{-1}) = (l^2 + h^4)^{1/4} |F^*(\mu, \sigma)|^{-1/2}$$

Here $k_{x,v}$ are the feedback coefficients with respect to x and v and the expressions $K_x = k_x|x|$, $K_v = k_v|v|$ have the meaning of feedback coefficients with respect to the unit vectors n_x, n_v respectively. Note that $K_{x,v}$ depend on two arguments: c and Φ_x^2, which are found by measurements and calculations using the elementary formulae: $c = (x, v)|x|^{-1}|v|^{-1}$, $\Phi_x^2 = x^2(x^2 + v^4)^{-1}$.

Expressions (5.5) are obtained from (5.4) by making the substitutions $l = |x|$, $h = |\upsilon|$, $c = (x, \upsilon)|x|^{-1}|\upsilon|^{-1}$, that is, by making the change $x^0 \to x$, $\upsilon^0 \to \upsilon$ and $t \to t - t_0$, $t_0 \to t$, that is, $t \to 0$. The Bellman function T in (5.5) has the meaning of the optimal time from the current phase point $x(t) = x$, $\upsilon(t) = \upsilon$. In the approach described, the main computational and analytical difficulties are encountered in constructing and analysing the functions $\mu*$, $\sigma*$, T.

To find the unknowns σ and μ, it is necessary to solve a system of transcendental equations presented in the following closed form

$$
\begin{aligned}
& f_x^2 / F^2 = \Phi_x^2, \quad f_{xv} / (|f_x||f_v|) = c \\
& F^2 = f_x^2 + f_v^4, \quad f_x^2 = a_\zeta^2\mu^2 + 2a_\zeta a_\eta \mu\sigma + a_\eta^2, \quad f_v^2 = b_\zeta^2\mu^2 + 2b_\zeta b_\eta\mu\sigma + b_\eta^2 \\
& f_{xv} = a_\zeta b_\zeta \mu^2 + (a_\zeta b_\eta + a_\eta b_\zeta)\mu\sigma + a_\eta b_\eta \\
& a_\zeta = -\mu^{-3}((\mu + 3\sigma)a + \mu + (3\sigma^2 - 1)b) \\
& a_\eta = b_\zeta = (a + \sigma b)/\mu^2, \quad b_\eta = -b/\mu, \quad a = (\mu^2 - 2\mu\sigma + 1)^{1/2} - 1 \\
& b = \ln((\mu - \sigma + (\mu^2 - 2\mu\sigma + 1)^{1/2})/(1 - \sigma))
\end{aligned}
\tag{5.6}
$$

Suppose that $\omega = (\mu - \sigma + (\mu^2 - 2\mu\sigma + 1)^{1/2})/(1 - \sigma)$, where ω is a new unknown parameter, where $\omega \geqslant 1$ and, moreover, the additional condition $\omega \leqslant 1/(1 - \mu)$ must be satisfied when $0 \leqslant \mu < 1$. Then

$$
\sigma = (\omega^2 - 2\mu\omega - 1)(\omega - 1)^{-2} \tag{5.7}
$$

We substitute (5.7) into the first of the equations of system (5.6) and obtain an equation of the fourth degree in μ which enables one to express μ in terms of ω and thereby reduce the problem to the solution of a single transcendental equation in ω. The corresponding expressions are extremely long and cannot be presented here.

By using computer algebra and numerical methods, a complete solution of the synthesis problem has been constructed: optimal control using the feedback principle and an expression for the Bellman function.

Families of graphs (curves 1–4) for the feedback coefficients $K_x(c, \Phi_x^2)$, $K_\upsilon(c, \Phi_x^2)$ with respect to the unit vectors of the position x and velocity υ respectively are shown in Figs 1 and 2. The variable c, $|c| \leqslant 1$ is taken as the argument and Φ^2_x as the parameter of the family, which takes the values $\Phi_x^2 = 0.1$, 0.2, 0.4 and 0.9, respectively. The small inset graphs in Figs 1 and 2 show the behaviour of the functions close to the value $c = -1$ and practically correspond to motion in a state of retardation along a straight line to the origin of coordinates.

A family of curves $F^{-1/2}(c, \Phi^2_x)$ for the above-mentioned values of c and Φ^2_x is shown in Fig. 3. The Bellman function $T = (l^2 + h^4)^{1/4} F^{-1/2}$ is constructed from it. The computational algorithm which

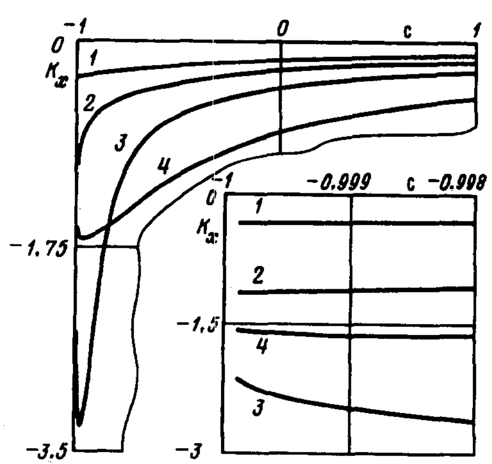

Fig. 1.

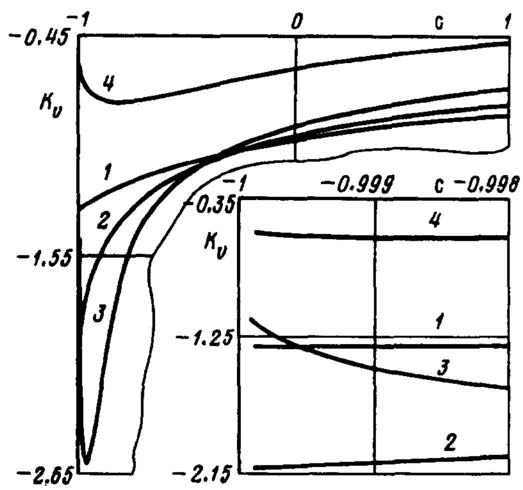

Fig. 2.

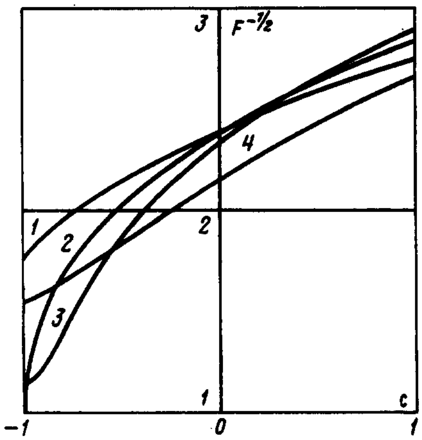

Fig. 3.

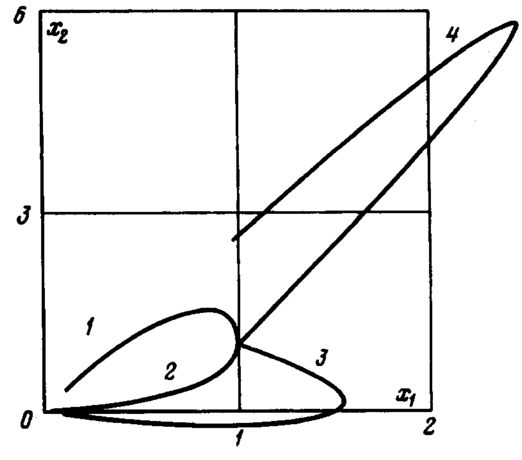

Fig. 4.

has been constructed enables one to obtain graphs for arbitrary, fairly frequent values of the parameter of the family Φ_x^2 (see the examples in the following section).

6. EXAMPLES

Trajectories are shown with the numbers 1–4 in Fig. 4 and, with the same numbers, Fig. 5 shows the corresponding relations between the components of the velocity which are obtained by using the approach developed above to solve the problem of a "soft landing" for the system

$$\ddot{x} = u, \quad x \in R^2, \quad |u| \leq 1, \quad x(0) = (1, 1) \tag{6.1}$$

for different initial values of the velocity.

The integration was performed using the standard Runge–Kutta method of the fourth order of accuracy with a variable step size and a control term in the Ingland form [9]. The average time for a calculation was about 30 min using an IBM PC AT 486 DX 40 computer and was highly dependent on the final accuracy which was required. Here, neither the initial C++ program nor the resulting machine code was subjected to optimization and the memory capacity required did not exceed 128 kB. The time required for a calculation can be significantly reduced by using standard optimization possibilities in conjunction with additional memory to store the intermediate results.

It should be noted that the equations describing the motion become highly degenerate in the case when the optimal control is directed precisely into the origin of coordinates. On account of this, the calculation was carried out up to the instant when the magnitude of c differed from −1 by not more than 10^{-5}. In all the cases considered, the subsequent motion reduces to a retardation where $\Phi_x^2 \to 0.2$, $c \to -1$. The values indicated are only attained

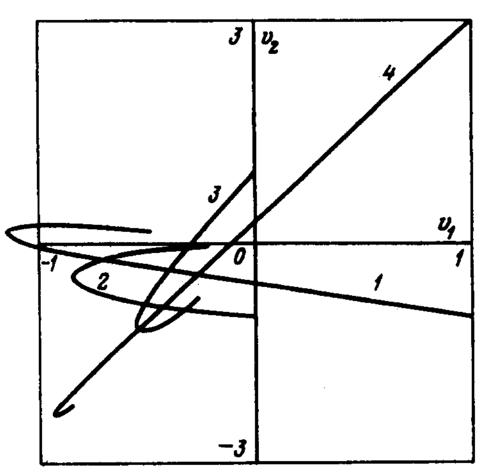

Fig. 5.

at the terminal point. There is probably no point in any subsequent increase in the accuracy in the majority of practical problems since, in practice, it is necessary to take account of the effect of various perturbations. In addition, there are a large number of methods which are specially designed to eliminate small errors. In the context of this theory, an increase in accuracy can be achieved by reducing the maximum step size in the integration, by using a more advanced technique to integrate ordinary differential equations and, also, by using different asymptotic forms.

We will now note a few facts.

1. A second solution which satisfies the Pontryagin maximum principle never arose during the course of the numerical experiments although uniqueness was not proved analytically.

2. A trajectory in which the initial velocity is directed exactly away from the origin of coordinates is obviously a "scattering" trajectory: in the case of close directions, trajectories which move away from it will be optimal. Mathematically, it is not possible to substantiate this observation at a given instant.

3. As the initial velocity increases, a rapid increase in ω is observed at the start of the motion. For instance, in the case of curve 4, the value of ω reaches magnitudes of the order of 10^6. This suggests the possibility of constructing the corresponding asymptotic form.

These observations show possible areas for subsequent analytic investigations.

Moreover, the results obtained enable one to obtain a numerical solution of the problem of a "soft landing" using comparatively small computational resources.

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